Generalized persistent homologies

Multidimensional persistent homology

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Outline

Review on shape comparison

Review on 1D persistent homology

Multidimensional Persistence

The persistence space

Discrete vs Continuous setting

What's going on

Shape comparison

- It is the task of evaluating similarities between given objects in a scene/dataset/image/video sequence.
- Useful in shape recognition/retrieval/classification:
  Given a query shape S, does the repository contain an object equal/similar/of the same class as S, in spite of
  - different view-point
  - different size or scale
  - translations and rotations
  - other deformations

Shape comparison pipeline

- Direct comparison in the shape space
  - a distance D is defined on the shape space

$D\Big($  ,  $\Big) = ?$

  - distance D gives a dissimilarity assessment among shapes
  - hard to compute

Shape comparison pipeline

- Comparison via signatures
  - A transform takes shapes to shape descriptors, or signatures

$$\text{Shapes space} \longrightarrow \text{Signatures space}$$

$$\longmapsto$$

    - compact representations of shapes
    - usually not sufficient to reconstruct the studied object
    - sufficient to identify an object as member of some class
  - a distance d is defined on the signatures space

$$d\left(\quad , \quad \right) = ?$$

  - easy to compute
  - ideally, signature distance = shape distance
  - in reality, signature distance $\leq$ shape distance

The category of shapes

Shapes are considered w.r.t. properties described by functions:

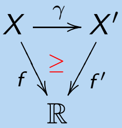
Objects

Pairs (X, f):

- X is a triangulable topological space
- $f : X \to \mathbb{R}$ is a continuous function

Morphisms

A morphism between two objects $(X, f), (X', f')$, is a continuous function $\gamma : X \to X'$ s. t. $f(x) \geq f'(\gamma(x))$ for all $x \in X$:



Direct comparison in the shapes category

Natural pseudo-distance

$$D\left((X,f),(Y,g)\right) = \begin{cases} \inf_{h\in H(X,Y)} \max_{P\in X} |f(P) - g\circ h(P))|, \\ +\infty \qquad\qquad \text{if} \;\; H(X,Y)=\emptyset, \end{cases}$$

$H(X,Y)$ being the set of all the homeomorphisms between X and Y.

[P. Frosini, M. Mulazzani: Size homotopy groups for computation of natural size distances, Bull. of the Belgian Math. Soc. - Simon Stevin, 6 (1999), 455-464]

Review on shape comparison

Review on 1D persistent homology

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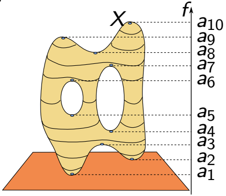
What's going on

From shapes to filtrations

- X a triangulable subspace of $\mathbb{R}^m$.
- $f : X \to \mathbb{R}$ a continuous function *with finitely many homological critical values* $a_1 < a_2 < \ldots < a_r$.
- For $s_0 < s_1 < \ldots < s_r$ s.t. $s_{i-1} < a_i < s_i$ set

$$X_i = f^{-1}((-\infty, s_i]).$$



- We obtain a filtration:

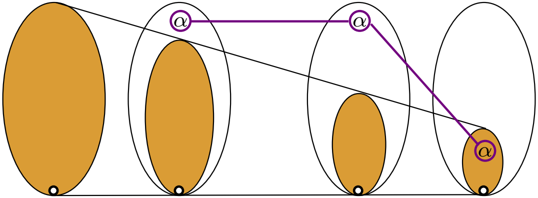
$$\emptyset = X_0 \hookrightarrow \ldots \hookrightarrow X_i \hookrightarrow \ldots \hookrightarrow X_j \hookrightarrow \ldots X_r = X$$

From topological spaces to vector spaces

- Passing to homology we obtain a sequence of homomorphisms:

$$0 = H_*(X_0) \to \ldots \to H_*(X_i) \to \ldots \to H_*(X_j) \to \ldots H_*(X_r) = H_*(X)$$

- Measure the lifespan of homology classes along the filtration:



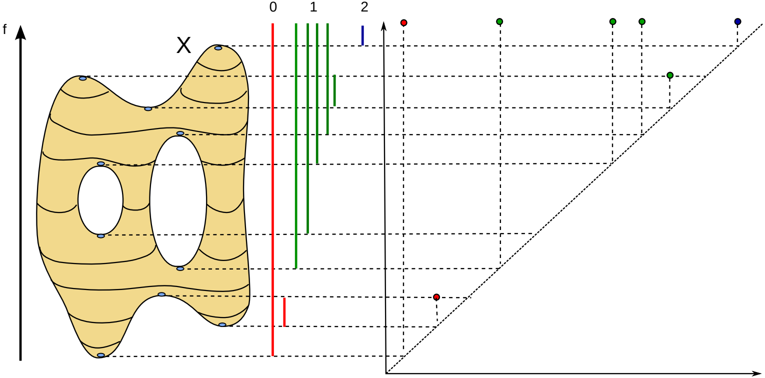
$$\ldots \to H_k(X_{i-1}) \to H_k(X_i) \to \ldots \to H_k(X_{j-1}) \to H_k(X_j) \to \ldots$$

α is born at X_i and dies at X_j

- *Persistent Betti Numbers (PBNs)*: for $u < v$, $\beta_f(u, v) = \mathrm{rk}\,\iota^{u,v}$.

Persistence diagrams (underlying idea)

- Encode the birth level i and the death level j of a homology class by a point (i, j)



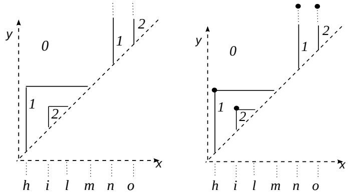
Persistence diagram (formally)

Multiplicities of points:

- $$\mu_f(x,y) = min_{\varepsilon>0} \quad \beta_f(x+\varepsilon, y-\varepsilon) - \beta_f(x-\varepsilon, y-\varepsilon) + \\ -\beta_f(x+\varepsilon, y+\varepsilon) + \beta_f(x-\varepsilon, y+\varepsilon)$$
- $\mu_f(x,\infty) = min_{\varepsilon>0,y} \quad \beta_f(x+\varepsilon, y) - \beta_f(x-\varepsilon, y)$

Definition

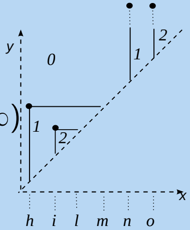
$(x,y) \in \mathrm{dgm}(f)$ iff $\mu_f(x,y) > 0$.



From a persistence diagram to PBNs

Fundamental lemma

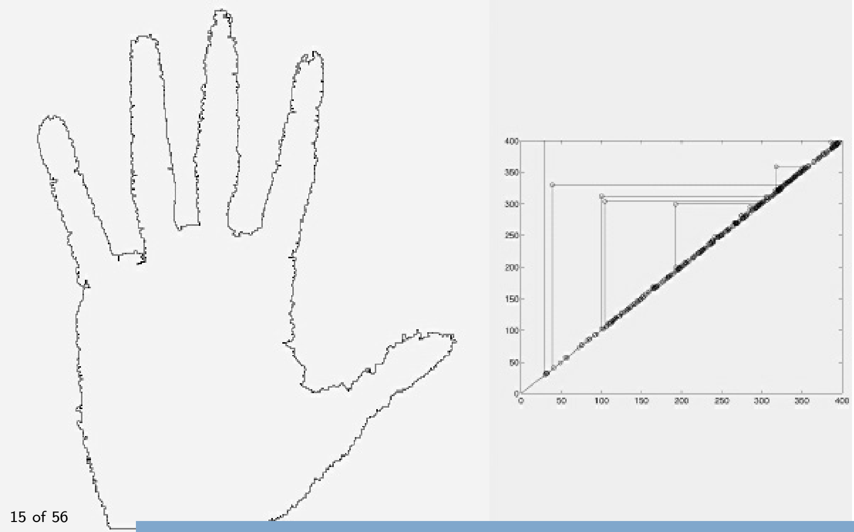
$$\beta_f(u,v) = \sum_{u' \le u, v' > v} \mu_f(u',v') + \sum_{u' \le u} \mu_f(u', \infty)$$



Caveat: Equality holds *almost everywhere*. It holds everywhere with Čech homology.

[Frosini – L.: Size functions and formal series, Appl. Algebra Engrg. Comm. Comput., **12**(4), 327–349 (2001)]
[Cohen-Steiner et al.: Stability of persistence diagrams, Discrete Comput. Geom., **37**(1), 103–120 (2007)]
[Cerri et al.: Betti numbers in multiD persistent homology are stable, Math. Meth. Appl. Sc., **36**, 1543–1557 (2013)]

The bottleneck distance of persistence diagrams



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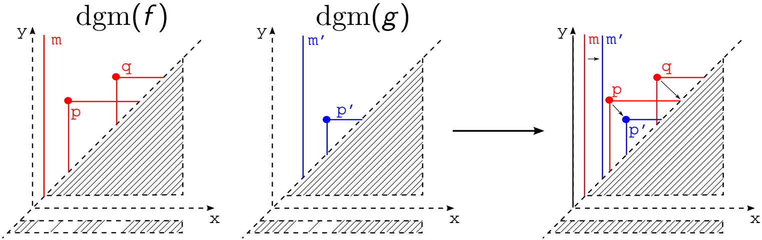

$d_B\left(\mathrm{dgm}(f), \mathrm{dgm}(g)\right) = \min_\gamma \max_{q \in \mathrm{dgm}(f)} \|q - \gamma(q)\|_\infty$ with γ any bijection between $\mathrm{dgm}(f)$ and $\mathrm{dgm}(g)$.

Stability theorem w.r.t. function perturbations

$$d_B\left(\mathrm{dgm}(f), \mathrm{dgm}(g)\right) \leq \|f - g\|_\infty$$

[Cohen-Steiner et al.: Stability of persistence diagrams, Discrete Comput. Geom., **37**(1), 103–120 (2007)]
[Chazal et al.: Proximity of persistence modules and their diagrams, Proc. SCG'09, 237–246 (2009)]
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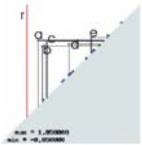
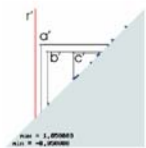
An optimal lower bound for D

D ( , ) $\rightsquigarrow d_B$ ( , )

Corollary

$d_B(\mathrm{dgm}(f), \mathrm{dgm}(g)) \leq D((X, f), (Y, g))$.

Theorem

Let d be a distance between persistence diagrams for H_0 such that the stability property holds: $d_B\left(\mathrm{dgm}(f), \mathrm{dgm}(g)\right) \leq \|f - g\|_\infty$.
Then $d \leq d_B$.

[d'Amico et al: Natural pseudo-distance and optimal matching between reduced size functions, Acta Appl.Math. 2010]

Motivation for multiD persistence

It is very desirable to obtain useful and computable summaries of the evolution of topology in situations where

- there is naturally more than one persistence parameter

- topology destroying noise needs to be smoothed out

The category of persistence modules

Define the category $\mathcal{M}$ of n-persistence modules:

- Objects are family of modules together with homomorphisms

$$\mathbf{M} = (\{M_u\}_{u \in \mathbb{R}^n}, \ \ \{\iota_M(u,v) : M_u \to M_v\}_{u \preceq v \in \mathbb{R}^n})$$

  such that, for all $u \preceq v \preceq w \in \mathbb{R}^n$, with $u = (u_i) \preceq v = (v_i)$ iff $u_i \leq v_i$,

$$\iota_M(u,w) = \iota_M(v,w) \circ \iota_M(u,v), \qquad \iota_M(u,u) = \mathrm{id}_{M_u}$$

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- Morphisms from $\mathbf{M}$ to $\mathbf{N}$ are collections of homomorphisms $\mathbf{h} = (h_u : M_u \to N_u)_{u \in \mathbb{R}^n}$ such that, for all $u \leq v \in \mathbb{R}^n$, $\iota_N(u,v) \circ h_u = h_v \circ \iota_M(u,v)$:

$$\begin{array}{ccc} M_v & \xrightarrow{h_v} & N_v \\ {\scriptstyle \iota_M(u,v)} \uparrow & \circlearrowleft & \uparrow {\scriptstyle \iota_N(u,v)} \\ M_u & \xrightarrow[h_u]{} & N_u \end{array}$$

From shapes to persistence modules

Consider the following categories:

- $\mathcal{C}$: the category of shapes,
- $\mathcal{M}$: the category of persistence modules.

We want to define a functor

$$\mathcal{C} \longrightarrow \mathcal{M}$$

From shapes to persistence modules

Consider the following categories:

- $\mathcal{C}$: the category of shapes,
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We want to define a functor

$$\mathcal{C} \longrightarrow \mathcal{M}$$

To this end we introduce an intermediate category $\mathcal{F}$: the category of filtrations.

The category of filtrations

Define the category $\mathcal{F}$:

- Objects are families of nested spaces $(X_u)_{u\in\mathbb{R}^n}$ with inclusions $i_{u,v}\colon X_u \hookrightarrow X_v$ whenever $u \leq v \in \mathbb{R}^n$.
- Morphisms are families $(\gamma_u)_{u\in\mathbb{R}^n}$ of maps $\gamma_u : X_u \to X'_u$ such that if $u \leq v \in \mathbb{R}^n$ $i'_{u,v} \circ \gamma_u = \gamma_v \circ i_{u,v}$, that is

$$\begin{array}{ccc} X_u & \xrightarrow{\gamma_u} & X'_u \\ {\scriptstyle i_{u,v}}\Big\downarrow & \circlearrowleft & \Big\downarrow{\scriptstyle i'_{u,v}} \\ X_v & \xrightarrow{\gamma_v} & X'_v \end{array}$$

The functor $F : \mathcal{C} \to \mathcal{F}$ from shapes to filtrations

In the category $\mathcal{C}$ of shapes (X, f) with $f : X \to \mathbb{R}^n$, and $u \in \mathbb{R}^n$, denote $X_u = \bigcap_{i=1}^{n} f_i^{-1}((-\infty, u_i])$

If $u \leq v \in \mathbb{R}^n$, there is the inclusion $i_{u,v} : X_u \hookrightarrow X_v$

If $\gamma : (X, f) \to (X', f')$ is a morphism in $\mathcal{C}$, the restriction of γ, $\gamma_u : X_u \to X'_u$ is a morphism in $\mathcal{F}$.

The functor $F : \mathcal{C} \to \mathcal{F}$ from shapes to filtrations

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If $\gamma : (X, f) \to (X', f')$ is a morphism in $\mathcal{C}$, the restriction of γ, $\gamma_u : X_u \to X'_u$ is a morphism in $\mathcal{F}$.

Define $F : \mathcal{C} \to \mathcal{F}$ by

- $F(X, f) = \left((X_u)_{u \in \mathbb{R}^n} \right)$
- $F(\gamma) = (\gamma_u)_{u \in \mathbb{R}^n}$

The persistent homology functor

Definition

The *i-th persistent homology functor* is the composite functor

$$H \circ F : \mathcal{C} \xrightarrow{F} \mathcal{F} \xrightarrow{H} \mathcal{M}$$

where H is the ordinary homology functor and F is the filtration functor.

[Chazal et al.: Proximity of persistence modules and their diagrams, Proc. SCG'09, 237–246 (2009)]
[Carlsson – Zomorodian: The theory of multidimensional persistence, Discr. Comput. Geom. **42**(1) (2009) 71–93]
[Lesnick: The theory of the interleaving distance on multidimensional persistence modules, Found. Comput. Math.]

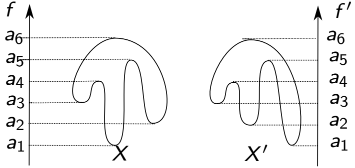
Properties of the persistent homology functor

- surjective on objects and on morphisms
  - For homology coefficients in $\mathbb{Q}$ or in $\mathbb{Z}/p\mathbb{Z}$ for some prime p, for every persistence module $\mathbf{M}$ there exists a CW-complex X and a continuous function $f : X \to \mathbb{R}^n$ such that $H_i \circ F(X, f) \cong \mathbf{M}$.
  - Let $\mathbf{h} : \mathbf{M} \to \mathbf{M}'$ be a homomorphism of persistence modules. Then there exist $(X, f), (X', f')$ and a continuous map $\gamma : X \to X'$ in $\mathcal{C}$ such that $H_i \circ F(X, f) \cong \mathbf{M}$, $H_i \circ F(X', f') \cong \mathbf{M}'$, and $H_i \circ F(\gamma) \cong \mathbf{h}$, for $i \in \mathbb{N}$ and for homology coefficients in $\mathbb{Q}$ or in $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

[Lesnick: The theory of the interleaving distance on multidimensional persistence modules, Found. Comput. Math.]

Properties of the persistent homology functor

- not full:



$$H_0 \circ F(X, f) \cong H_0 \circ F(X', f)$$

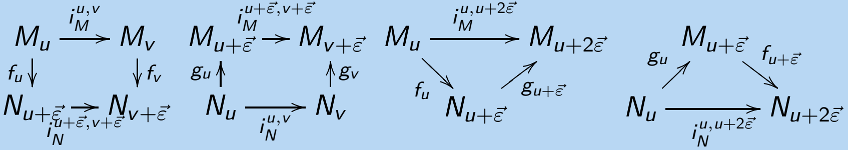
but $\nexists \gamma : (X, f) \to (X', f')$ such that $H_0 \circ F(\gamma)$ is an isomorphism

[Cagliari et al: Persistence modules, shape description, and completeness, Proc. CTIC 2012]

Interleavings of persistence modules

ε-interleaving

M, **N** are ε-inteleaved, $\varepsilon > 0$, if there exists f, g such that the following diagrams commute for $u \preceq v \in \mathbb{R}^n$:



[Chazal et al.: Proximity of persistence modules and their diagrams, Proc. SCG'09, 237–246 (2009)]

The interleaving distance of persistence modules

Interleaving distance

$$d_I(\mathbf{M}, \mathbf{N}) = \inf\{\varepsilon \geq 0 : \mathbf{M}, \mathbf{N} \text{ are } \varepsilon\text{-inteleaved}\}$$

Theorem

$d_I(H \circ F(X, f), H \circ F(Y, g)) \leq D((X, f), (Y, g))$

Theorem

d_I is an "optimal lower bound" for $D((X, f), (Y, g))$.

[Lesnick: The theory of the interleaving distance on multidimensional persistence modules, Found. Comput. Math.]

Persistent homology over $\mathbb{N}^n$ and graded $k[x_1, x_2, \ldots, x_n]$-modules

The correspondence α such that $\alpha(\mathbf{M}) = \bigoplus_{v \in \mathbb{N}^n} M_v$ where the action of $x^v = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$ is given by shifting elemnets of the module up in the gradation defines an equivalence of categories between the category of persistence modules of finite type over k and the category of finitely generated non-negatively graded modules over $k[x_1, x_2, \ldots, x_n]$.

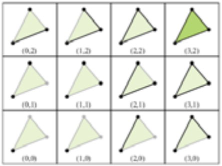


Figure 2: A bifiltration of a triangle.

$$\begin{array}{ccccccc}
k^2 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
k^2 & \longrightarrow & k^3 & \longrightarrow & k & \longrightarrow & k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
k & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k
\end{array}$$

$$M_{(0,1)} = \langle a, b \rangle,$$
$$x_1 x_2 (a - b) = 0$$

[Carlsson – Zomorodian: The theory of multidimensional persistence, Discr. Comput. Geom. **42**(1) (2009) 71–93]

Classification of persistence modules for $n = 1$

- For $n = 1$, persistence modules are completely classified by persistence diagrams:

$$\mathbf{M} \cong \bigoplus_{i=1}^{n} \sum^{\alpha_i} k[t] \oplus \bigoplus_{j=1}^{m} \sum^{\beta_j} k[t]/(t^{\gamma_j})$$

PBNs	$\Leftrightarrow$	persistence diagram/barcode	$\Leftrightarrow$	persistence module
		$d_B(\mathrm{dgm}\mathbf{M}, \mathrm{dgm}\mathbf{N})$	$=$	$d_I(\mathbf{M}, \mathbf{N})$

[Zomorodian – Carlsson: Computing Persistent Homology, Discrete Comput. Geom, **33** (2005) 249–274]

Classification of persistence modules for $n > 1$

- For $n > 1$, no discrete and complete invariant exist:
  - example: for lines $l_1 \neq l_2 \neq l_3$ in k^2, the isomorphism classes of the persistence module

$$
\begin{array}{ccccc}
k^2/l_1 & \rightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
k^2 & \rightarrow & k^2/l_2 & \rightarrow & 0 \\
{\scriptstyle Id}\uparrow & & \uparrow & & \uparrow \\
k^2 & \longrightarrow & k^2 & \rightarrow & k^2/l_3
\end{array}
$$

can be enumerated by lines in k^2 (i.e. $\mathbb{P}^1(k)$).

[Carlsson – Zomorodian: The theory of multidimensional persistence, Discr. Comput. Geom. **42**(1) (2009) 71–93]

Known invariants of persistence modules

- $\xi_0(\mathbf{M})$ = multiset in $\mathbb{R}^n$ giving locations where generators are born
- $\xi_1(\mathbf{M})$ = multiset in $\mathbb{R}^n$ giving locations where relations between generators are born
- $\xi_2(\mathbf{M})$ = multiset in $\mathbb{R}^n$ giving locations where relations between relations are born
- $\xi_i(\mathbf{M})$, $i = 0, \dots n$

$$\mathbf{M} = \begin{array}{ccccccc} & & \vdots & & \vdots & & \\ \cdots & & C & \xrightarrow{f} & D & \rightarrow & \cdots \\ & & {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle g} & & \\ \cdots & & A & \xrightarrow[\alpha]{} & B & \rightarrow & \cdots \\ & & \vdots & & \vdots & & \end{array}$$

$$\xi_i = H_i(A \overset{(\alpha,\beta)}{\to} B \oplus C \overset{f-g}{\to} D)$$
$$\xi_0 = \mathrm{coker}(f-g)$$
$$\xi_1 = \ker(f-g)/\mathrm{im}(\alpha,\beta)$$
$$\xi_2 = \ker(\alpha,\beta)$$

[Chacholski-Scolamiero: Private communication]

Known invariants of persistence modules

- $\xi_0(\mathbf{M})$ = multiset in $\mathbb{R}^n$ giving locations where generators are born NOT STABLE
- $\xi_1(\mathbf{M})$ = multiset in $\mathbb{R}^n$ giving locations where relations between generators are born NOT STABLE
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Known invariants of persistence modules

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- $\xi_2(\mathbf{M})$ = multiset in $\mathbb{R}^n$ giving locations where relations between relations are born NOT STABLE
- $\xi_i(\mathbf{M})$, $i = 0, \ldots n$ NOT STABLE
- PBNs or rank invariant:

$$\beta_{\mathbf{M}} : \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u \prec v\} \to \mathbb{N} \cup \{\infty\}$$

$$\beta_{\mathbf{M}}(u, v) = \mathrm{rk}\,\iota_M(u, v)$$

PBNs stability w.r.t. noisy functions

- $L : u = s\vec{m} + b$ line in $\mathbb{R}^n$ parametrized by $s \in \mathbb{R}$

PBNs stability w.r.t. noisy functions

- $L : u = s\vec{m} + b$ line in $\mathbb{R}^n$ parametrized by $s \in \mathbb{R}$
- $\mathbf{M}_L$ restriction of the n-dim p.f.d. persistence module $\mathbf{M}$ to L: $(M_L)_s = M_u$, $\iota^{s,s'} = \iota^{u,u'}$, with $u = s\vec{m} + b$, $u' = s'\vec{m} + b$ in L

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- $d_{match}(\beta_{\mathbf{M}}, \beta_{\mathbf{N}}) = \sup_{L: u = s\vec{m}+b} \min_i m_i \cdot d_B(\mathrm{dgm}\mathbf{M}_L, \mathrm{dgm}\mathbf{N}_L)$

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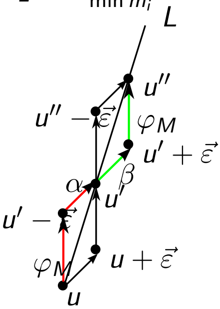
- $\mathbf{M}$, $\mathbf{N}$ are ε-interleaved $\implies$ $\mathbf{M}_L$, $\mathbf{N}_L$ are $\frac{\varepsilon}{\min m_i}$-interleaved



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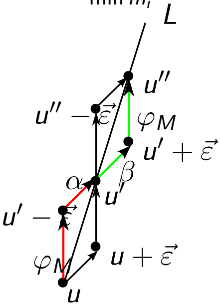
- $\mathbf{M}$, $\mathbf{N}$ are ε-interleaved $\implies$ $\mathbf{M}_L$, $\mathbf{N}_L$ are $\frac{\varepsilon}{\min m_i}$-interleaved
- $d_{match}(\beta_{\mathbf{M}}, \beta_{\mathbf{N}}) \leq d_I(\mathbf{M}, \mathbf{N})$



PBNs stability w.r.t. noisy functions

- $L : u = s\vec{m} + b$ line in $\mathbb{R}^n$ parametrized by $s \in \mathbb{R}$
- $\mathbf{M}_L$ restriction of the n-dim p.f.d. persistence module $\mathbf{M}$ to L: $(M_L)_s = M_u$, $\iota^{s,s'} = \iota^{u,u'}$, with $u = s\vec{m} + b$, $u' = s'\vec{m} + b$ in L
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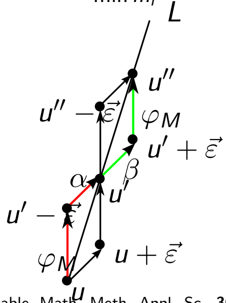


- $d_{match}(\beta_{\mathbf{M}}, \beta_{\mathbf{N}}) \leq d_I(\mathbf{M}, \mathbf{N})$
  PBNs are STABLE

[Cerri et al.: Betti numbers in multiD persistent homology are stable, Math. Meth. Appl. Sc., **36**, 1543–1557 (2013)]

PBNs stability w.r.t. noisy domains

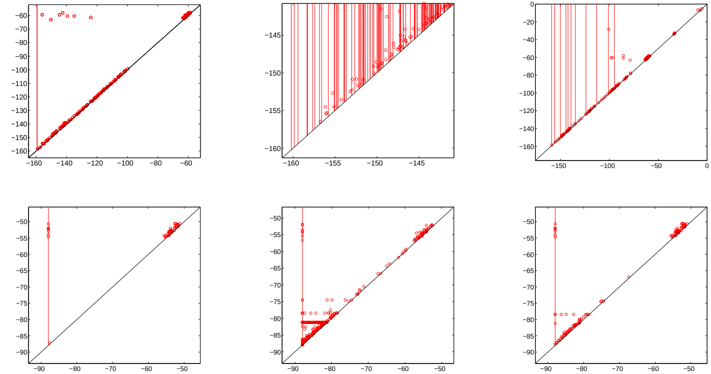
Theorem

Let K_1, K_2 be non-empty closed subsets of a triangulable subspace X of $\mathbb{R}^n$. Let $d_{K_1}, d_{K_2} : X \to \mathbb{R}$ be their respective distance functions. Moreover, let $\vec{\varphi}_1, \vec{\varphi}_2 : X \to \mathbb{R}^k$ be vector-valued continuous functions. Then, defining $\vec{\Phi}_1, \vec{\Phi}_2 : X \to \mathbb{R}^{k+1}$ by $\vec{\Phi}_1 = (d_{K_1}, \vec{\varphi}_1)$ and $\vec{\Phi}_2 = (d_{K_2}, \vec{\varphi}_2)$, the following inequality holds:

$$d_{match}\left(\beta_{\vec{\Phi}_1}, \beta_{\vec{\Phi}_2}\right) \leq \max\{\delta_H(K_1, K_2), \|\vec{\varphi}_1 - \vec{\varphi}_2\|_\infty\}.$$

[Frosini – L.: Persistent Betti numbers for a noise tolerant shape-based approach to image retrieval, Pattern Recogn. Lett., **34** (2013), 1320-1321.]

PBNs stability w.r.t. noisy domains: examples



PBNs internal stability

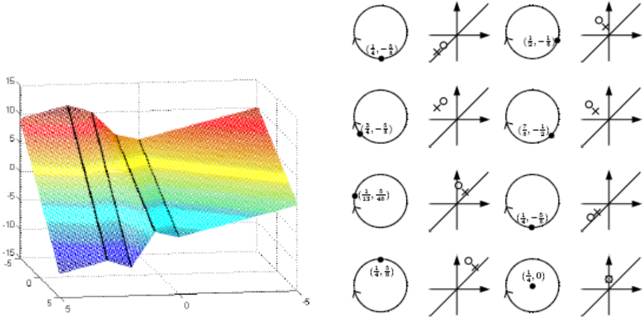
For a fixed persistence module **M**, the 1-D persistence diagram does not change too much when we perturb the line:

Let **M** be a p.f.d. persistence module for which there exist $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ such that $\varphi_{\mathbf{M}}(u, u')$ is an isomorphism for every $u, u' \in \mathbb{R}^n$ with $c \prec u \preceq u'$. Let $L : u = s\vec{m} + b$ and $L' : u = s\vec{m}' + b'$. There exist constants $K, C > 0$ such that $\mathbf{M}_L$ and $\mathbf{M}_{L'}$ are η-interleaved, and therefore $d_B(\mathbf{M}_L, \mathbf{M}_{L'}) \leq \eta$, with

$$\eta = \frac{K \cdot \|\vec{m} - \vec{m}'\|_\infty + C \cdot \|b - b'\|_\infty}{\min m_i \cdot \min m'_i}.$$

[Cerri et al.: Betti numbers in multiD persistent homology are stable, Math. Meth. Appl. Sc., **36**, 1543–1557 (2013)]

Monodromy in multiD PBNs

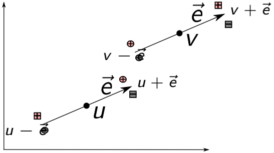


[Cerri et al: A study of monodromy in the computation of multidimensional persistence, In: Proc. DGCI 2013]

Definitions

For $f : X \to \mathbb{R}^k$, $(u, v) \in \mathbb{R}^k \times \mathbb{R}^k$, $u \prec v$

- $\mu_f(u,v) = \min_{\vec{e} \succ \vec{0},} \begin{array}{l} \beta_f(u+\vec{e}, v-\vec{e}) - \beta_f(u-\vec{e}, v-\vec{e}) \\ -\beta_f(u+\vec{e}, v+\vec{e}) + \beta_f(u-\vec{e}, v+\vec{e}). \end{array}$



- $\mu_f(u,\infty) = \min_{\vec{e} \succ \vec{0},\, v} \beta_f(u+\vec{e}, v) - \beta_f(u-\vec{e}, v)$.
- The *persistence space* is the multiset of all points p such that $\mu_f(p) > 0$, with their multiplicity, union the points of $\Delta = \{(u,v) \in \mathbb{R}^k \times \mathbb{R}^k : u \preceq v \text{ and } \exists i \text{ s.t } u_i = v_i\}$, with infinite multiplicity.

Distance from Δ and persistence of a point

- The distance of a point $p = (u, v) \in \mathbb{R}^k \times \mathbb{R}^k$ with $u \prec v$ to Δ is

$$\inf_{q \in \Delta} \|p - q\|_\infty = \min_{i=1,\ldots,k} \frac{v_i - u_i}{2}.$$

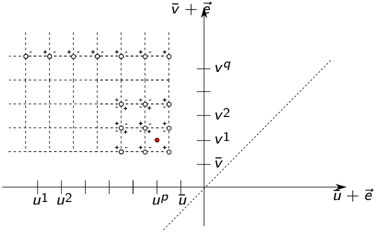
- The *persistence* of a point $p = (u, v) \in \mathbb{R}^k \times \mathbb{R}^k$ with $u \prec v$ and multiplicity $\mu_f(p) > 0$ is

$$\mathrm{pers}(p) = \min_{i=1,\ldots,k} v_i - u_i.$$

Fundamental lemma

For every $\bar{u} \prec \bar{v} \in \mathbb{R}^k$ and for every $\vec{e} \succ 0 \in \mathbb{R}^k$, it holds that

$$\beta_f(\bar{u}, \bar{v}) = \sum_{\substack{u \preceq \bar{u},\ v \succ \bar{v} \\ \bar{u} - u,\, v - \bar{v} \in \langle \vec{e} \rangle}} \mu_f(u, v) + \sum_{\substack{u \preceq \bar{u} \\ \bar{u} - u \in \langle \vec{e} \rangle}} \mu_f(u, \infty).$$



Stability Theorem

Let $f, g : X \to \mathbb{R}^k$ be continuous functions. Then

$$d_H(\mathrm{Spc}(f), \mathrm{Spc}(g)) \leq \max_{x \in X} \|f(x) - g(x)\|_\infty,$$

where the Hausdorff distance between $\mathrm{Spc}(f)$ and $\mathrm{Spc}(g)$ is

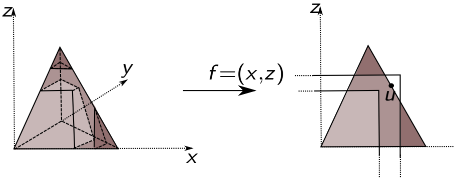
$$\max\{ \sup_{p \in \mathrm{Spc}(f)} \inf_{q \in \mathrm{Spc}(g)} \|p - q\|_\infty, \sup_{q \in \mathrm{Spc}(g)} \inf_{p \in \mathrm{Spc}(f)} \|p - q\|_\infty \}.$$

Link to homological critical values

Definition

$u \in \mathbb{R}^k$ is a *homological critical value* of $f \in C^0(M, \mathbb{R}^k)$ if, $\forall \varepsilon > 0$ small enough, $\exists u', u'' \in \mathbb{R}^k$ s.t.

- $u' \preceq u \preceq u''$,
- $\|u' - u\|_\infty \leq \varepsilon$, $\|u'' - u\|_\infty \leq \varepsilon$,
- $H_*(M_{u'} \hookrightarrow M_{u''})$ is not an isomorphism.



Theorem

- If $\mu_f(u, v) > 0$, then u and v are homological critical values of f.
- If $\mu_f(u, \infty) > 0$, then u is a homological critical value for f.

Link to Pareto critical values

Definition

$u \in \mathbb{R}^k$ is a *Pareto critical value* of $f = (f_1, \dots, f_k) \in C^r$ if $\exists p \in M$ s.t.

- $f(p) = u$,
- $0 \in \mathrm{Conv}(\nabla f_1(p), \dots, \nabla f_k(p))$

Theorem

- If $\mu_f(u, v) > 0$, then $j(u)$ and $j(v)$ are Pareto critical values of $j \circ f$ for some linear projection $j : \mathbb{R}^k \to \mathbb{R}^h$, $h < k$.
- If $\mu_f(u, \infty) > 0$, then $j(u)$ is a Pareto critical value of $j \circ f$ for some linear projection $j : \mathbb{R}^k \to \mathbb{R}^h$, $h < k$.

Persistence spaces and reduction to 1D-filtrations

The one-parameter filtration of X obtained by sweeping the line $L : u = s\vec{e} + b$ corresponds to the sublevel sets of

$$F_{(u,v)} = \max_i \left\{ \frac{f_i(x) - b_i}{e_i} \right\}.$$

The following statements hold:

- For $u \prec v$ with $(u,v) = (s\vec{e} + b, t\vec{e} + b)$, it holds that $(u,v) \in \mathrm{Spc}(f)$ iff $(s,t) \in \mathrm{dgm}(F_{(u,v)})$;
- For $u = s\vec{e} + b$, it holds that $(u,\infty) \in \mathrm{Spc}(f)$ iff $(s,\infty) \in \mathrm{dgm}(F_{(u,v)})$.

Moreover, $\frac{\mathrm{pers}(u,v)}{\mathrm{pers}(s,t)} = \min_i e_i$.

Comparison with persistence diagrams

Persistence Diagram

- defined via multiplicities
- Representation Thm
- for $f \in C^0$, link with homological critical values
- for $f \in C^\infty$, link with critical values

Persistence Space

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[Cerri–L.: Hausdorff stability of persistence spaces, Found. Comput. Math.]

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[Cerri–L.: Hausdorff [illegible] Found. Comput. Math.]

Experimental results: PHOG

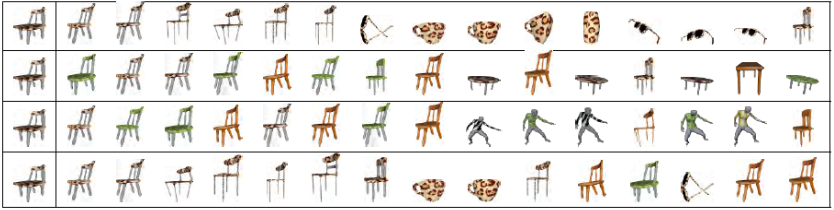
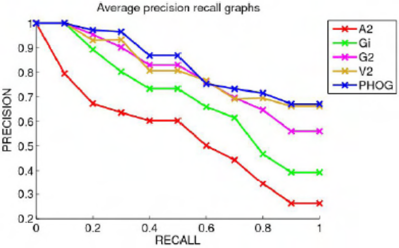


Figure 7: *From high to below: retrieved items with respect to: LAB, hybrid, geometric and combined description.*



Continuous vs discrete setting

- Sub-level set filtrations are those for which *stability results* hold:
  $\forall f, f' : X \to \mathbb{R}^k$ continuous functions, $d(\beta_f, \beta_{f'}) \leq \|f - f'\|_\infty$.

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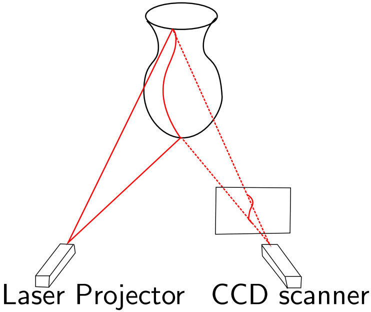
- Discrete filtrations are those actually used in computations:



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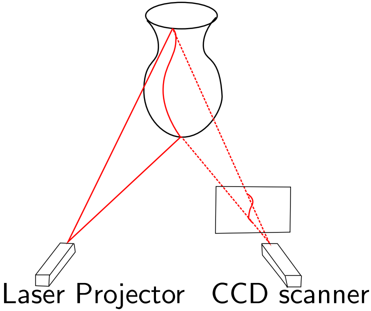

Stable comparison of rank invariants obtained from discrete data?

From discrete to continuous filtrations

Question: How to extend $\varphi : \mathcal{V}(K) \to \mathbb{R}^k$ to a continuous function $K \to \mathbb{R}^k$ so that its sub-level set filtration coincides with $\{K_\alpha\}_{\alpha \in \mathbb{R}^k}$?

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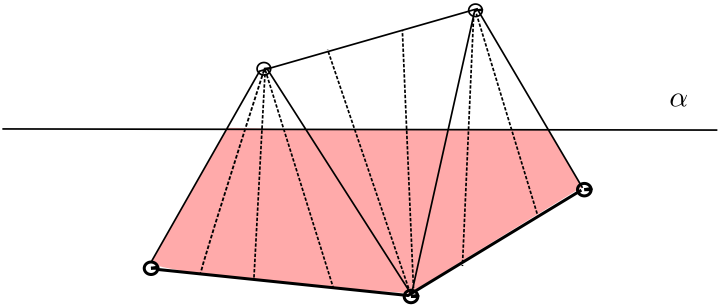
Answer: 1-D persistence: use linear interpolation [Morozov, 2008]



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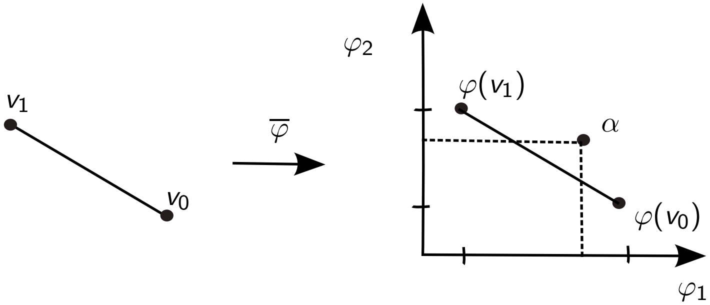
Answer: Multi-D persistence:

linear interpolation yields topological aliasing



Topological Aliasing: numerical experiments

	Original	Linear int.	% Diff
	cat0 vs. cat0-tran1-1		
H_1	0.046150	0.040576	-**13.737185**
H_0	0.225394	0.207266	-**8.746249**
	cat0-tran1-2 vs. cat0-tran2-1		
H_1	0.034314	0.029188	-**17.562012**
H_0	0.208451	0.204511	-**1.926547**
	cat0-tran2-1 vs. cat0-tran2-2		
H_1	0.045545	0.037061	-**22.891989**
H_0	0.212733	0.208097	-**2.227807**

Axis-wise linear interpolation

- Given any $\sigma \in \mathcal{K}$, set $\mu(\sigma) = max\{\varphi(v) \mid v$ is a vertex of $\sigma\}$.
- Use induction to define $\varphi^{\urcorner} : K \to \mathbb{R}^k$ on σ and a point $w_\sigma \in \sigma$ s.t.
  - For all $x \in \sigma$, $\varphi^{\urcorner}(x) \preceq \varphi^{\urcorner}(w_\sigma) = \mu(\sigma)$;
  - $\varphi^{\urcorner}$ is linear on any line segment $[w_\sigma, y]$ with $y \in \partial\sigma$.

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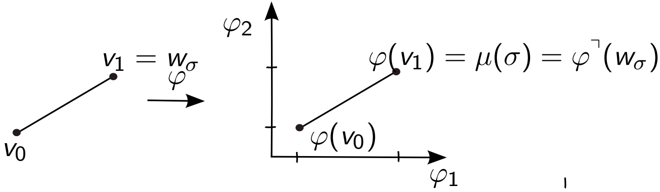

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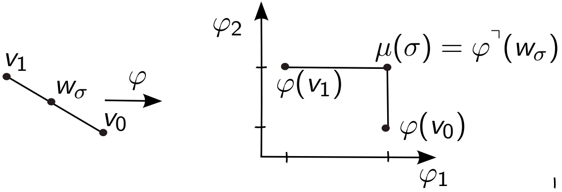

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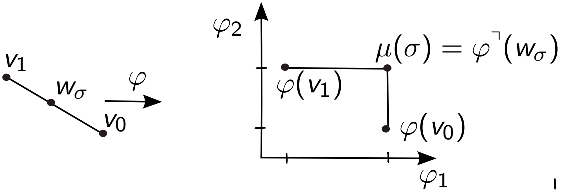

Theorem

For any $\alpha \in \mathbb{R}^k$, K_α is a strong deformation retract of $K_{\varphi^{\urcorner} \preceq \alpha}$.

Bridging stability to discrete persistent homology

- X and Y homeomorphic triangulable spaces (real objects);
- $f : X \to \mathbb{R}^k$,$g : Y \to \mathbb{R}^k$ continuous (real measurements);
- $\mathcal{K}'$ and $\mathcal{L}'$ simplicial complexes with $|\mathcal{K}'| = K$, $|\mathcal{K}'| = L$ (approximated object);
- $\tilde{\varphi} : K \to \mathbb{R}^k$, $\tilde{\psi} : L \to \mathbb{R}^k$ continuous (approximated

Theorem: If two homeomorphisms $\xi : K \to X$, $\zeta : L \to Y$ exist s.t.

$$\|\tilde{\varphi} - f \circ \xi\|_\infty \leq \epsilon/4, \quad \|\tilde{\psi} - g \circ \zeta\|_\infty \leq \epsilon/4$$

then, for any sufficiently fine subdivision $\mathcal{K}$ of $\mathcal{K}'$ and $\mathcal{L}$ of $\mathcal{L}'$,

$$|d_{match}(\beta_f, \beta_g) - d_{match}(\beta_\varphi, \beta_\psi)| \leq \epsilon,$$

$\varphi : \mathcal{V}(\mathcal{K}) \to \mathbb{R}^k$, $\psi : \mathcal{V}(\mathcal{L}) \to \mathbb{R}^k$ being restrictions of $\tilde{\varphi}$ and $\tilde{\psi}$.

Sketch of the proof

- $\exists\delta > 0$ s.t. $\max\{\operatorname{diam}\sigma \mid \sigma \in \mathcal{K}$ or $\sigma \in \mathcal{L}\} < \delta \implies$

$$|d_{match}(\beta_{\tilde{\varphi}}, \beta_{\tilde{\psi}}) - d_{match}(\beta_{\varphi^{\urcorner}}, \beta_{\psi^{\urcorner}})| < \epsilon/2.$$

[Cavazza et al: Comparison of persistent homologies for vector functions: from continuous to discrete and back, Comput. Math. Appl. **66** (2013), 560-573]

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$$|d_{match}(\beta_{\tilde{\varphi}}, \beta_{\tilde{\psi}}) - d_{match}(\beta_{\varphi^{\neg}}, \beta_{\psi^{\neg}})| < \epsilon/2.$$

- $\beta_{\varphi} = \beta_{\varphi^{\neg}}$, $\beta_{\psi} = \beta_{\psi^{\neg}}$.

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-

$$\begin{aligned} d_{match}(\beta_f, \beta_g) &\leq d_{match}(\beta_f, \beta_{f\circ\xi}) + d_{match}(\beta_{f\circ\xi}, \beta_{\tilde{\varphi}}) + d_{match}(\beta_{\tilde{\varphi}}, \beta_{\tilde{\psi}} \\ &+ d_{match}(\beta_{\tilde{\psi}}, \beta_{g\circ\zeta}) + d_{match}(\beta_{g\circ\zeta}, \beta_g) \quad \square \end{aligned}$$

.

[Cavazza et al: Comparison of persistent homologies for vector functions: from continuous to discrete and back, Comput. Math. Appl. **66** (2013), 560-573]

On-going research

- Carlsson et al (Stanford): coordinatization of the set of persistence modules
  - construction of an infinite family of functions, each of which takes as input any persistence module and outputs a nonnegative number
- Lesnick et al (IMA): Exact computation of multi-dimensional matching distance
- Chacholski et al (KTH): New stable invariants of persistence modules
- Frosini et al (UniBO): Monodromy in PBNs
- Landi (UniMORE) and Cerri (CNR-IMATI): bottleneck distance for persistence spaces
- Kaczynski et al (Univ. Sherbrooke): multidimensional persistent homology and discrete Morse theory

To know more

Applied Algebraic Topology Research Network:

http://www.ima.umn.edu/topology/

Applied Topology:

http://appliedtopology.org/

To know more

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THANK YOU FOR YOUR ATTENTION!